GOSTS Iterated Forcing

Martin’s Axiom and More General Theory

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In general, forcing takes a preorder $\mathbb{P} \in V$ and gives a filter $G$ intersecting all dense subsets of $\mathbb{P}$ in $V$ with various nice properties telling us $V[G]$ satisfies nice things. Often however,

- In showing these properties, we don’t need $G$ to intersect *all* dense sets in $V$.
- We want $V[G]$ to be similar to $V$ (e.g. don’t collapse cardinals) which is often given by properties of $\mathbb{P}$.

If we only enforce our $G$ to intersect a few dense subsets of sufficiently nice $\mathbb{P}$, it may be that $G \in V$ already so that $V = V[G]$ and we get lots of nice forcing results for free!
Defining Martin’s Axiom

**Definition**

For $\kappa$ an ordinal, $\text{MA}(\kappa)$ is the statement

\[
\text{for every ccc preorder } \mathbb{P} \text{ and family } \mathcal{D} \text{ of open, dense sets of } \mathbb{P}, \\
\text{if } |\mathcal{D}| \leq \kappa \text{ then there is a } G \ \mathbb{P}\text{-generic over } \mathcal{D}.
\]

$\text{MA}$ is the statement $\forall \kappa < 2^{\aleph_0} \ \text{MA}(\kappa)$.

This is in some sense (but not really) the max we could possibly hope for:

- $\text{ZFC} \vdash \text{MA}(\omega) + \neg \text{MA}(2^{\aleph_0})$.
- So clearly $\text{ZFC} + \text{CH} \vdash \text{MA}$, hence we are more interested in $\text{MA} + \neg \text{CH}$.
- If we strengthen $\text{MA}$ to include both ccc and non-ccc preorders, this modified $\text{MA}(\aleph_1)$ is able to kill $\aleph_1$-Aronszajn trees (which provably exist), and thus this strengthening is *equivalent* to $\text{CH}$.
Theorem

\[ \text{ZFC} \vdash \text{MA}(\aleph_0) + \neg \text{MA}(2^{\aleph_0}). \]

Proof.

- For \( \text{MA}(\aleph_0) \), if \( \mathcal{D} = \{ D_n : n \in \omega \} \) is a countable collection of dense open sets, then we just take \( p_0 \in D_0 \), extend to \( p_1 \in D_1 \) and so on. The result has \( G = \{ p_n : n \in \omega \} \) with \( G \cap D_n \neq \emptyset \). So just take the upward closure to get a filter.

- To see \( \neg \text{MA}(2^{\aleph_0}) \), consider Cohen forcing, adding a single subset of \( \omega \). \( \text{Add}(\aleph_0, 1) \) is ccc so we can pretty easily calculate that there are only \( |\text{Add}(\aleph_0, 1)|^{\aleph_0} = 2^{\aleph_0} \)-many antichains and thus \( 2^{\aleph_0} \)-many dense sets in \( V \). \( \text{MA}(2^{\aleph_0}) \) would imply there is a generic over all dense sets meaning a full generic \( G \in V \), which is impossible. \( \square \)
If we are going to force $\text{MA} + \langle 2^{\aleph_0} = \kappa \rangle$, we need to know what information $\text{MA}$ tells us about $2^{\aleph_0}$ so that we can start out with those properties on $\kappa$. When looking at $\text{MA} + \neg \text{CH}$, we should expect *some* information about what $2^{\aleph_0}$ can be. It turns out that the only real restriction comes from

$$ZFC + \text{MA} \vdash \langle 2^{\aleph_0} \text{ is regular} \land 2^{<2^{\aleph_0}} = 2^{\aleph_0} \rangle.$$ 

Showing this just amounts to finding and using the right preorders and dense subsets.

- In the end, rather than showing directly that $2^{\aleph_0}$ is regular, we will show $2^\kappa = 2^{\aleph_0}$ for all $\kappa < 2^{\aleph_0}$.
- This shows by König’s theorem that $\kappa < \text{cof}(2^\kappa) = \text{cof}(2^{\aleph_0})$ for $\kappa < 2^{\aleph_0}$. 
For us, the right preorder is almost disjoint forcing. Recall:

- Two sets $A, B \subseteq \omega$ are almost disjoint iff $|A \cap B| < \aleph_0$.
- An almost disjoint family is a collection $\mathcal{A} \subseteq \mathcal{P}(\omega)$ where any two elements of $\mathcal{A}$ are almost disjoint.
- We can show there’s an almost disjoint family of size $2^{\aleph_0}$.

The preorder $\text{Adp}(\mathcal{A}, X)$ for $X \subseteq \mathcal{A}$ essentially adds a subset of $\omega$ that codes membership in $X$. Given that we may take $|\mathcal{A}| = \kappa < 2^{\aleph_0}$, MA would say we can code any particular subset $X \subseteq \kappa$ by a subset of $\omega$ and thus $2^\kappa \leq 2^{\aleph_0}$ for $\kappa < 2^{\aleph_0}$.

In the end, we will have $X = \{A \in \mathcal{A} : g \cap A \text{ is infinite}\}$ where $g$ is our added subset. To approximate the characteristic function of $g$, take $\text{Adp}(\mathcal{A}, X)$ to be the set of $p$ such that

1. $p$ is a partial function from $\omega$ to $2$ (codes a subset of $\omega$);
2. For all $A \in X$, $A \cap \text{dom}(p)$ is finite (ensuring we build up $g$ properly); and
3. $p^{-1}\{1\}$ is finite (ensuring $g \cap A$ is finite for $A \notin X$).

Note that $p \in \text{Adp}(\mathcal{A}, X)$ might be infinite: we only require (the set coded by) $p$’s intersection with elements of $X$ to be finite. Outside of these sets, $p$ may contain lots of 0s.
We also order this in the natural way: $p \leq q$ iff $p \supseteq q$.

**Lemma**

Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be an almost disjoint family of sets with $X \subseteq \mathcal{A}$. Therefore $\text{Adp}(\mathcal{A}, X)$ is ccc.

**Proof.**

- Note that if $p \perp q$ then they disagree on some $n \in \omega$. Since the only two choices are 0 and 1, $p^{-1}\{1\} \neq q^{-1}\{1\}$.
- So if $W \subseteq \text{Adp}(\mathcal{A}, X)$ is an antichain, $U = \{p^{-1}\{1\} : p \in W\}$ is also an antichain with $|U| = |W|$.
- But each $p^{-1}\{1\}$ is a finite subset of $\omega$ and therefore $|W| = |U| \leq \aleph_0$.

This means that we can actually use this preorder with MA.
As said before, $\text{Adp}(\mathcal{A}, X)$ codes the subset $X \subseteq \mathcal{A}$ by a subset $g \subseteq \omega$. Hence if we force with this for all $X \in \mathcal{P}(\mathcal{A})$, we get $2^{\lvert \mathcal{A} \rvert} = 2^{\aleph_0}$.

**Theorem**

Let $\kappa < 2^{\aleph_0}$. Therefore $\text{MA}(\kappa)$ implies $2^\kappa = 2^{\aleph_0}$ and so $\text{cof}(2^{\aleph_0}) > \kappa$. In particular, $\text{MA}$ implies $2^{\aleph_0}$ is regular.

The steps to showing this are mostly: for $G$ generic,

- Show $\text{Adp}(\mathcal{A}, X)$ does what we want, meaning $\bigcup G$ is the characteristic function of a subset $g \subseteq \omega$ coding membership in $X$:

$$X = \{A \in \mathcal{A} : \lvert g \cap A \rvert = \aleph_0\}.$$

- Show that we only use $\kappa$-many dense sets in the argument.
- Conclude from $\text{MA}$ that $2^\kappa \leq 2^{\aleph_0}$.
Proof.

Let $\kappa < 2^{\aleph_0}$ be arbitrary. There’s an a.d. family of size $2^{\aleph_0}$, so by reducing to a subset, we may consider an a.d. family of size $\kappa$, $\mathcal{A} = \{A_\alpha : \alpha < \kappa\}$. Let $X \subseteq \mathcal{A}$ be arbitrary.

We will give $|\mathcal{A}|^2 \cdot \aleph_0$-many dense sets of $\text{Adp}(\mathcal{A}, X)$ such that if $G$ is generic over them, then $g = (\bigcup G)^{-1}\{1\} \subseteq \omega$ codes membership in $X$. For $n \in \omega$, $A_\alpha \in X$, $A_\beta \notin X$, consider

$$D_\beta = \{p : A_\beta \subseteq \text{dom}(p)\}$$

$$E^n_\alpha = \{p : n \leq |A_\alpha \cap p^{-1}\{1\}|\}.$$

- $D_\beta$ is dense (just output 0 for $n \in A_\beta$).
- $E^n_\alpha$ is dense (just add finitely many 1s).
- Any $G$ generic over these codes a subset $g = (\bigcup G)^{-1}\{1\}$.

These $|\mathcal{A}|^2 \cdot \aleph_0 = |\mathcal{A}| = \kappa$-many dense sets will give $A \in X$ iff $g \cap A$ is infinite for $A \in \mathcal{A}$. 
Proof.

For $A_\alpha \in X$, $A_\beta \in \mathcal{A} \setminus X$, $n < \omega$;

\[
D_\beta = \{p : A_\beta \subseteq \text{dom}(p)\}
\]
\[
E^n_\alpha = \{p : n \leq |A_\alpha \cap p^{-1}\{1\}|\}.
\]

• $g = (\bigcup G)^{-1}\{1\}$.

• If $A_\alpha \in X$, $G \cap E^n_\alpha \neq \emptyset$ for every $n < \omega$ so that $|g \cap A_\alpha| \geq n$ for every $n < \omega$ and hence $g \cap A_\alpha$ is infinite.

• If $A_\beta \notin X$, $G \cap D_\beta \neq \emptyset$. So any $p \in G \cap D_\beta$ has already decided all values of the characteristic function of $g$ on $A_\beta$. As a condition, $p^{-1}\{1\}$ is finite and hence $p$ must have output a bunch of 0s: $g \cap A_\beta$ is finite.

This establishes $A \in X$ iff $g \cap A$ is infinite for $A \in \mathcal{A}$. 
Proof.

Now we may use $\text{MA}(\kappa)$ and think about $2^\kappa$. Any $Y \subseteq \kappa = |\mathcal{A}|$ yields a ccc preorder $\text{Adp}(\mathcal{A}, \{A_\alpha : \alpha \in Y\})$. By $\text{MA}(\kappa)$, there’s a $g_Y \subseteq \omega$ where $Y = \{\alpha < \kappa : |g_Y \cap A_\alpha| = \aleph_0\}$. This gives an injection from $\mathcal{P}(\kappa)$ to $\mathcal{P}(\omega)$ by $Y \mapsto g_Y$. Thus $2^\kappa = 2^{\aleph_0}$.

Thus if $\text{MA}$ holds, $\kappa < \text{cof}(2^\kappa) = \text{cof}(2^{\aleph_0})$ holds for all $\kappa < 2^{\aleph_0}$ and therefore $\text{cof}(2^{\aleph_0}) \geq 2^{\aleph_0}$.

One may think of this as saying that $\text{MA}$ has bumped up the size of the continuum by adding subsets of $\omega$ for each subset of $\kappa < 2^{\aleph_0}$. \qed
In proving $\text{Con}(\text{ZFC} + \text{MA} + \neg \text{CH})$ (from $\text{Con}(\text{ZFC})$), the naïve idea is just to force with all ccc preorders. This is too many preorders to actually force with practically. So it’s useful to just deal with smaller preorders.

**Lemma**

$\text{MA}(\kappa)$ is equivalent to $\text{MA}(\kappa)$ restricted to preorders of size $\leq \kappa$. In other words, we have generics over $\leq \kappa$-many dense subsets of arbitrary ccc preorders iff we have generics over $\leq \kappa$-many dense subsets just for ccc preorders of size $\leq \kappa$. 
Lemma

**MA(κ) is equivalent to MA(κ) restricted to preorders of size ≤ κ.**

Proof.

Let $\mathbb{P}$ be a ccc of arbitrary size and $\mathcal{D}$ a $\leq \kappa$-sized family of dense sets. Consider the skolem hull $\mathcal{H} = \text{Hull}_{\mathbb{P}}(\emptyset)$ of $\mathbb{P}$ in the language $\mathcal{L}_\mathcal{D} = \{\leq, D : D \in \mathcal{D}\}$ (where $D^\mathbb{P}$ is membership in $D$).

- $\mathcal{H}$ is an elementary submodel of $\mathbb{P}$ of size $|\mathcal{H}| \leq \kappa$.
- Density is $\mathcal{L}_\mathcal{D}$-definable—“$\forall x \exists y \ (y \leq x \land D(y))$”—so by elementarity, $D^{\mathcal{H}} = D \cap \mathcal{H}$ is dense for each $D \in \mathcal{D}$.
- Incompatibility is $\mathcal{L}_\mathcal{D}$-definable—“$\neg \exists r \ (r \leq p, q)$”—so by elementarity, $\mathcal{H}$ is still ccc.

Thus if $G$ is $\mathcal{H}$-generic over $\{D \cap \mathcal{H} : D \in \mathcal{D}\}$, then $G \upharpoonright$ is $\mathbb{P}$-generic over $\mathcal{D}$. $\square$
At this point, we are more or less prepared to prove the consistency of $\text{MA} + \neg \text{CH}$. There are some formal issues to worry about, however, and I haven’t yet introduced the technology to deal with them.

The basic idea behind the proof is to force with some preorder $\mathcal{P} = \bigotimes_{\alpha < \kappa} \dot{Q}_\alpha$ to get $V[G] \models \text{MA}$. To show this, we want to have already forced with every ccc preorder in that if $Q, D \in V[G]$ both of size $\leq \kappa$, then $Q, D$ actually appears by some stage $V[G \upharpoonright \alpha]$.

But in $V$, $Q$ and $D$ are given by $\mathcal{P}$-names, not $\bigotimes_{\xi < \alpha} \dot{Q}_\xi$-names. How can we compare the two? We need a way to translate names, and this leads to the idea of various kinds of homomorphisms.
The natural way to translate conditions between preorders is with homomorphisms, that is order preserving (and 1-preserving) maps.

**Definition**

For preorders $\mathbb{P}, \mathbb{Q}$, $f : \mathbb{P} \rightarrow \mathbb{Q}$ is an *incompatibility homomorphism* iff it’s a homomorphism preserving incompatibility, i.e. for all $p, p' \in \mathbb{P}$;

- $f(1^\mathbb{P}) = 1^\mathbb{Q}$;
- $p \leq^\mathbb{P} p'$ implies $f(p) \leq^\mathbb{Q} f(p')$;
- $p \perp^\mathbb{P} p'$ implies $f(p) \perp^\mathbb{Q} f(p')$.

In addition, for $f$ an incompatibility homomorphism,

- $f$ is a *dense homomorphism* iff $f''\mathbb{P}$ is dense in $\mathbb{Q}$;
- $f$ is a *complete homomorphism* iff for any maximal antichain $\mathcal{A} \subseteq \mathbb{P}$, $f''\mathcal{A}$ is a maximal antichain of $\mathbb{Q}$.

We actually won’t need much of the discussion that follows for forcing MA, but it’s useful to know nonetheless.
We’ve seen examples of incompatibility homomorphisms before with limit stages of iterations: \( p \mapsto p \prec \langle \mathbb{1}, \mathbb{1}, \cdots \rangle \).

It’s not hard to show that every dense homomorphism is complete (mostly due to the fact that every dense set like \( f''\mathbb{P} \) contains a maximal antichain).

I will not focus so much on complete homomorphisms here, but they get their use from the following theorem, telling us that forcing with \( \mathbb{Q} \) already forces with \( \mathbb{P} \) and we translate to \( \mathbb{P} \) simply by taking the preimage of the generic.

**Theorem**

*If \( f : \mathbb{P} \to \mathbb{Q} \) is a complete homomorphism with \( \mathbb{P}, \mathbb{Q}, f \in V \), then for any \( G \mathbb{Q} \)-generic over \( V \), \( f^{-1}''G \) is \( \mathbb{P} \)-generic over \( V \) and \( V[f^{-1}''G] \subseteq V[G] \).*

(See notes for a proof.)
Dense embeddings are better than this because they allow us to go both directions: complete homomorphisms only let us go backwards. This gives the notion of forcing equivalence.

**Definition**

Two preorders $\mathbb{P}$, $\mathbb{Q}$ are *forcing equivalent* iff

- Every generic extension by $\mathbb{P}$, $V[G]$, is equal to a generic extension by $\mathbb{Q}$, $V[H]$; and
- Every generic extension by $\mathbb{Q}$, $V[H]$, is equal to a generic extension by $\mathbb{P}$, $V[G]$.

As a side note, this can be written in a first-order way without reference to the existence of generics (in particular, they’re forcing equivalent iff their boolean algebras corresponding to regular open sets are isomorphic.)
Theorem

If \( f : \mathbb{P} \to \mathbb{Q} \) is a dense embedding then \( \mathbb{P} \) and \( \mathbb{Q} \) are forcing equivalent. In fact,

1. For \( G \) \( \mathbb{P} \)-generic, \( (f"G)\uparrow \) is \( \mathbb{Q} \)-generic with \( V[G] = V[f"G\uparrow] \);

and

2. For \( H \) \( \mathbb{Q} \)-generic, \( f^{-1}"H \) is \( \mathbb{P} \)-generic with \( V[f^{-1}"H] = V[H] \).

Proof of (1).

It’s not difficult to show \( f"G\uparrow \) is a filter. For genericity, let \( D \subseteq \mathbb{Q} \) be dense and open. Consider \( f^{-1}"D \).

- \( f^{-1}"D \) is dense in \( \mathbb{P} \) (for \( p \in \mathbb{P} \), moving to \( \mathbb{Q} \), \( f(p) \) has an extension \( q \in D \). Since \( f"\mathbb{P} \) is dense, there’s an extension \( f(r) \leq^\mathbb{Q} q \) also in \( D \) because \( D \) is open. \( f(r) \leq^\mathbb{Q} f(p) \) implies \( r \leq^\mathbb{P} p \) as an embedding with \( r \in f^{-1}"D \).

- \( p \in G \cap f^{-1}"D \) yields \( f(p) \in f"G\uparrow \cap D \) so that \( f"G\uparrow \) is generic.
Theorem

If \( f : \mathbb{P} \to \mathbb{Q} \) is a dense embedding and \( G \) is \( \mathbb{P} \)-generic, then \( (f''G) \uparrow \) is \( \mathbb{Q} \)-generic with \( V[G] = V[f''G] \).

Proof of (1).

To show \( V[G] = V[f''G] \), clearly \( V[f''G] \subseteq V[G] \) since \( f, G, \mathbb{Q} \in V[G] \). The other direction follows from \( f \) being a complete homomorphism: recall the unproven result that if \( H \) is \( \mathbb{Q} \)-generic then \( f^{-1}''H \) is \( \mathbb{P} \)-generic with \( V[f^{-1}''H] \subseteq V[H] \). We merely take \( H = f''G \uparrow \) so \( f^{-1}''H = G \).

The proof of (2) is similar, only requiring us to check \( V[f^{-1}''H] = V[H] \), which is easy using (1) given that \( H = f''(f^{-1}''H) \uparrow \).
Forcing Equivalence

This allows us a lot of flexibility in how we define our forcing notions. For example, all of the following are forcing equivalent: (all ordered by \( \supseteq \) with max element \( \emptyset \))

- \( \text{Add}(\omega, 1) \), i.e. \( \{ p : \omega \to 2 : |p| < \aleph_0 \} \)
- \( <^{\omega} 2 \), i.e. partial functions with domain an entire natural number
- \( <^{\omega} \omega \), i.e. partial functions with domain an entire natural number
- \( \{ p : \omega \to \omega : |p| < \aleph_0 \} \)
- \( <^{\omega} [\omega] \) (ordered by end extension)

This also pretty easily allows us to show that forcing with preorders is the same as with posets by taking any choice function on the equivalence classes. (See proof in notes.)

**Theorem**

Let \( \mathbb{P} \) be a preorder. Write \( p \approx q \) iff \( p \leq_{\mathbb{P}} q \leq_{\mathbb{P}} p \), an equivalence relation. Therefore \( \mathbb{P}/\approx \) is a poset forcing equivalent to \( \mathbb{P} \) by any \( f : \mathbb{P}/\approx \to \mathbb{P} \) satisfying \( f([p]\approx) \approx p \).
Let’s get back on track:

- for proving $\text{Con}(\text{MA} + \neg \text{CH})$, we need to be able to translate names in an intelligible way, especially between iterations.
- The natural way to do this is with a way to translate the elements of the preorders, i.e. a homomorphism, changing $\langle \sigma, p \rangle$ to $\langle \sigma, f(p) \rangle$.
- If we recursively do this, this works. And in fact when $f$ is a dense embedding, we can say much more.

**Theorem**

Let $f : \mathbb{P} \rightarrow \mathbb{Q}$ be an incompatibility homomorphism. Therefore there is a function $T : V^\mathbb{P} \rightarrow V^\mathbb{Q}$ such that

1. for all $\mathbb{Q}$-generics $H$ and $\tau \in V^\mathbb{P}$, $T(\tau)_H = \tau_{f^{-1}''H}$;
2. if $f$ is a dense embedding, $G$ is $\mathbb{P}$-generic and $\tau \in V^\mathbb{P}$, then $\tau_G = T(\tau)_f''G\uparrow$; and
3. if $f$ is a dense embedding and $H$ is $\mathbb{Q}$-generic, then $V[H] = \{T(\tau)_H : \tau \in V^\mathbb{P}\}$.

In particular, if $f$ is a dense embedding, then $\mathbb{1}^\mathbb{P} \models \varphi(\tau)$ iff $\mathbb{1}^\mathbb{Q} \models \varphi(T(\tau))$ for every formula $\varphi$. 
So this tells us $T$ really does translate the names in the strongest possible sense. Luckily for us, the definition of $T$ is incredibly easy: $T(\emptyset) = \emptyset$ and recursively,

$$T(\tau) = \{\langle T(\sigma), f(p) \rangle : \langle \sigma, p \rangle \in \tau \}.$$ 

Showing that this works mostly is just a matter of chasing definitions and using some of the results above regarding forcing equivalence. (See notes for a proof.)

The basic idea behind this theorem is that (1) and (2) show that $f$ indeed can translate names that will be interpreted the same. (3) is useful in that it shows $T$ is “surjective” in the sense that it covers all the names used by a generic extension via $\mathbb{Q}$ even if not every name is in the image of $T$. 
Now we can force MA. Recall what we know about MA thus far:

- MA is equivalent to MA restricted to preorders of size $< 2^{\aleph_0}$.
- MA implies $2^{\aleph_0}$ is regular.
- MA implies $2^{<\aleph_0} = 2^{\aleph_0}$.

So we will want to start with some regular cardinal $\kappa > \aleph_1$ where $2^{<\kappa} = \kappa$ and in the end get $2^{\aleph_0} = \kappa$. The general idea to proceed is

- Enumerate all the ccc preorders of size $< \kappa$ we might get to in a $\kappa$-stage iteration. (This also requires $2^{<\kappa} = \kappa$)
- Force with these preorders with a finite support iteration.

We want a finite support iteration because (1) it preserves ccc-ness, and (2) some combinatorial results mostly about counting names and preserving cofinalities.
We begin with the following setup:

- \( V \models ZFC + GCH \) is a transitive model we can force over.
- \( \kappa \) is a regular, uncountable cardinal of \( V \).

The general argument is as follows:

1. for \( \theta < \kappa \), there are at most \( 2^\theta \)-many preorders of size \( \leq \theta \) (up to isomorphism).
2. By GCH (and really, just \( 2^{< \kappa} = \kappa \)), this means we can enumerate all preorders of size \( < \kappa \) by a sequence of length \( \kappa \):

\[
\{ Q : Q \text{ is a preorder } \land |Q| = Q < \kappa \} = \{ P_\alpha : \alpha < \kappa \}.
\]
3. We then force with these \( P_\alpha \)'s. And then repeat (1)–(3) \( \kappa \)-many times.
4. By reordering (and translating!) the names, we can force with these \( \kappa \times \kappa = \kappa \)-many posets in \( \kappa \) stages.
5. Then we confirm MA in the generic extension.
To proceed in this way, we need to confirm that we can keep doing this at every stage, this means we need $2^{<\kappa} = \kappa$ to continually hold. This is partially a result of things being ccc and choosing nice names.

To get on with the actual proof, we will define a finite support iteration $\bigstar_{\alpha < \kappa} \dot{\mathcal{Q}}_{\alpha}$, giving a ccc preorder in $V$ because we will require for each $\tilde{\xi} < \kappa$

$$1_{\tilde{\xi}} \models "\dot{\mathcal{Q}}_{\tilde{\xi}} \text{ is ccc} \land |\dot{\mathcal{Q}}_{\tilde{\xi}}| < \tilde{\kappa}"$$

(\ast)

Without loss of generality, we will work with preorders that are “standard” in the sense that $\mathcal{Q} = |\mathcal{Q}|$ is an ordinal.
Claim

If $\mathbb{P}$ is a ccc preorder with $|\mathbb{P}| \leq \kappa$ (in $V$) then $\Vdash \text{"}\exists \check{\kappa} \leq \check{\kappa}\text{"}$ (and thus $\Vdash \text{"}2^{\aleph_0} \leq \check{\kappa}\text{"}$)

Proof.

Count nice names: there are at most $|\mathbb{P}|^{\aleph_0 \cdot \theta} \leq \kappa^{< \kappa} = \kappa$-many nice names in $V$ for subsets of $\check{\theta}$ for any cardinal $\theta < \kappa$.

This will apply to us given that $(\ast)$ holds inductively.

Claim

Suppose $\ast_{\xi < \alpha} \dot{\mathbb{Q}}_{\xi}$ is a finite support iteration and $(\ast)$ holds for every $\xi < \alpha$. Therefore $\ast_{\xi < \alpha} \dot{\mathbb{Q}}_{\alpha}$ is ccc of size $\leq \kappa$ and therefore $\Vdash \text{"}2^{< \check{\kappa}} = \check{\kappa}\text{"}$.

The proof of this is pretty boring, just the result of counting names and using small antichains to limit the amount of names.
\[
\forall \xi < \kappa \; \Vdash_\xi \; \text{``} Q_\xi \text{ is ccc } \land |Q_\xi| < \tilde{\kappa} \text{''} \tag{\star} \\
\forall \alpha \leq \kappa \; \Vdash_\alpha \; \text{``} 2^{<\kappa} \leq \tilde{\kappa} \text{''} \tag{\star\star}
\]

Okay, so if \((\star)\) holds inductively, we get \((\star\star)\) and can continue to define the iteration. But how do we actually define this iteration? Remember the original process:

1. for \(\theta < \kappa\), there are at most \(2^\theta\)-many preorders of size \(\leq \theta\) (up to isomorphism).
2. By \((\star\star)\) (and really, just \(2^{<\kappa} = \kappa\)), this means we can enumerate all (canonical names for) ccc preorders of size \(< \kappa\) by a sequence of length \(\kappa\)
3. We then force with these. And then repeat (1)–(3) \(\kappa\)-many times.
\[ \forall \xi < \kappa \models \xi \models \text{“} \dot{\mathcal{Q}}_\xi \text{ is ccc } \land |\dot{\mathcal{Q}}_\xi| < \check{\kappa} \text{”} \quad (\ast) \]

\[ \forall \alpha \leq \kappa \models \text{“} 2^{<\check{\kappa}} \leq \check{\kappa} \text{”} \quad (\ast\ast) \]

So for \( \ast \xi < \alpha \dot{\mathcal{Q}}_\xi \) defined, let

\[ \models \text{“} \langle \dot{\mathcal{P}}_{\alpha, \beta} : \beta < \kappa \rangle \text{ enumerates the standard ccc preorders of size } < \check{\kappa} \text{”} \]

Now we can’t just force with these all at once, and we can’t just force with these linearly since then we’d end up with a \( \kappa \)-stage iteration immediately and even longer if we repeat in that way (iterations of length \( \kappa \), we might not have \( (\ast\ast) \) anymore).

So we just enumerate them relative to some bijection \( f : \kappa \to \kappa \times \kappa \): at stage \( \alpha < \kappa \) we want to force with \( \dot{\mathcal{P}}_{f(\alpha)} = \dot{\mathcal{P}}_{f_0(\alpha), f_1(\alpha)} \). To ensure this has been defined, we also want \( f_0(\alpha) \leq \alpha \) which is easy to arrange.
The issue here is that $\dot{P}_{f(\alpha)}$ is a $\star_{\xi<\alpha} \dot{Q}_{\xi}$-name, not a name of the later iteration $\star_{\xi<\alpha} \dot{Q}_{\xi}$. The name translation work before tells us we don’t actually care.

Recall that the map $\iota_{\beta,\alpha}$ from the $\beta$th stage iteration to the $\alpha$th stage iteration (adding a bunch of $1$s at the end) is an incompatibility homomorphism. Thus we have a map $T_{\beta,\alpha}$ from $\star_{\xi<\beta} \dot{Q}_{\xi}$-names to $\star_{\xi<\alpha} \dot{Q}_{\xi}$-names that gives the same interpretation at later stages. The name translation theorem before gives:

Claim

Let $\star_{\xi<\alpha} \dot{Q}_{\xi}$ be a finite support $\alpha$-stage iteration. Let $G \upharpoonright \alpha$ be $\star_{\xi<\alpha} \dot{Q}_{\xi}$-generic over $V$ and $\beta < \alpha$: Therefore $T_{\beta,\alpha}(\tau)_{G \upharpoonright \alpha} = \tau_{G \upharpoonright \beta}$ for $\tau$ a $\star_{\xi<\beta} \dot{Q}_{\xi}$-name.
GOSTS Iterated Forcing

James Holland

Martin’s Axiom
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Back to MA
Proving \text{Con}(MA + \neg CH)

Forcing MA + \neg CH

\forall \xi < \kappa \ 1_\xi \models \text{“} \dot{Q}_\xi \text{ is ccc } \wedge |\dot{Q}_\xi| < \check{k}\text{”}  \quad (*)

\forall \alpha \leq \kappa \ 1_\alpha \models \text{“} 2^{<\check{k}} \leq \check{k}\text{”}  \quad (**) 

Now I still haven’t told you what we’re forcing with, but translating the names makes this pretty clear: at stage \alpha in our iteration \bigstar_{\xi < \alpha} \dot{Q}_\xi, define \dot{Q}_\alpha to be \mathcal{T}_{f_0(\alpha), \alpha}(\dot{P}_{f(\alpha)}) if this is a (name for a) standard ccc preorder, and otherwise some other (name for a) standard ccc preorder.

- This defines \bigstar_{\xi < \alpha + 1} \dot{Q}_\xi.
- Taking finite supports defines the limit stages as direct limits.
- Considering \mathcal{P} = \bigstar_{\xi < \kappa} \dot{Q}_\xi, (**) tells us \mathcal{1}_\mathcal{P} \models \text{“} \mathcal{2}^{\aleph_0} \leq \kappa\text{”}.

To show \mathcal{1}_\mathcal{P} \models \text{“} \mathcal{2}^{\aleph_0} \geq \kappa\text{”}, we need to show the more difficult result that \mathcal{1}_\mathcal{P} \models \text{“} \text{MA}(\check{\theta})\text{”} for every \theta < \kappa.
Now we begin the actual meat of the proof: everything thus far has just been some formal bookkeeping that allows us to force will all of these $\dot{P}_{\alpha, \beta}$ for $\alpha, \beta < \kappa$. To show that $\mathcal{P} \models \text{“MA}(\check{\theta})\text{”,}$ we basically need to have already forced with all of our $\theta$-sized preorders.

- If $\mathcal{Q}, \mathcal{D} \in V[G]$ with $|\mathcal{Q}|, \mathcal{D} < \kappa$, we will get an $\alpha < \kappa$ with $\mathcal{Q}, \mathcal{D} \in V[G \upharpoonright \alpha]$.

- This isn’t free: although $\mathcal{P}$ is the direct limit of previous iterations, $V[G]$ is never the direct limit of the $V[G \upharpoonright \alpha]$ for $\alpha < \kappa$.

- Instead we use the regularity of $\kappa$. If $q^* \leq q$, let $\alpha_{q^*, q} = \sup \text{sprt}(p)$ be minimal among the $p \in \mathcal{P}$ forcing $p \models \text{“} \check{q}^* \leq \check{\mathcal{Q}} \check{q} \text{”}$ (remember $q^*, q \in \mathcal{Q}$ are ordinals).

- Let $\alpha = \sup_{q^* \leq q} \alpha_{q^*, q}$. Since $\kappa$ is regular and $|\mathcal{Q}| < \kappa$, $\alpha < \kappa$.

- But then all the relevant forcing happens below $\alpha$: in $V[G \upharpoonright \alpha]$,

$$\leq^{\mathcal{Q}} = \{ \langle q^*, q \rangle \in \theta \times \theta : \exists p \in G \upharpoonright \alpha (\imath_{\alpha, \kappa}(p) \models \text{“} \check{q}^* \leq \check{\mathcal{Q}} \check{q} \text{”}) \}.$$  

- So $\mathcal{Q} \in V[G \upharpoonright \alpha]$, and a similar result works for $\mathcal{D}$ by considering the graph $\{ \langle \alpha, D \rangle : \alpha \in D \in \mathcal{D} \}$. 
But then a name for $\mathcal{Q}$ appears somewhere on our list of ccc preorders to force with: $\mathcal{Q} = (\dot{\mathbb{P}}_{\alpha,\beta})_{G \uparrow \alpha}$ for some $\beta < \kappa$.

- Since we did some good bookkeeping with $f : \kappa \rightarrow \kappa \times \kappa$ a bijection, we forced with $\mathcal{Q}$ (or really $T_{\alpha,\alpha'}(\dot{\mathbb{P}}_{\alpha,\beta})$) at some stage $\alpha' < \kappa$.

- Since the dense sets of $V[G \uparrow \alpha']$ contains at least the dense sets of $\mathcal{D}$, in $V[G \uparrow \alpha' + 1]$, $G_{\alpha'}$ is $\mathcal{Q}$-generic over the dense sets of $V[G \uparrow \alpha']$ and in particular over $\mathcal{D}$.

- Being generic over this is upward absolute: $G_{\alpha'} \in V[G]$ witnesses $\text{MA}(\theta)$ for $\mathcal{Q}$, $\mathcal{D}$.

So, $V[G] \vDash \text{MA}(\theta)$ for every $\theta < \kappa$ implying $\kappa \leq 2^{\aleph_0}$ and so $V[G] \vDash \text{``MA} + \kappa = 2^{\aleph_0}\text{''}$. As $G$ was arbitrary, $\mathbb{P}$ forces this.

This completes the proof.